

Microscopic identification of dissipative modes in relativistic field theories

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Received September 4, 2014; Revised April 9, 2015; Accepted April 13, 2015; Published May 22, 2015

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 We present an argument to support the existence of dissipative modes in relativistic field theories. For an $O(N)$ φ^4 theory in spatial dimension $1 \leq d \leq 3$, the two-point function of φ is shown to develop a pole of the form $p_0 \sim i\Gamma^{-1}(\mathbf{p}^2 + m^2)$ at small energy p_0 and momentum \mathbf{p} , with a nonzero finite coefficient or relaxation constant, Γ , when evaluated in the two-particle irreducible (2PI) framework at the next-leading order (NLO) of $1/N$ expansion. In contrast, an NLO calculation in the one-particle irreducible (1PI) framework fails to give a nonzero Γ . The dissipative mode emerges from multiple scattering of a particle with other particles, which is appropriately treated in the 2PI–NLO calculation through the resummation of secular terms to improve the long-time behavior of the two-point function. Assuming that this slow dissipative mode survives at the critical point, one can identify the dynamic critical exponent z for the two-point function as $z = 2 - \eta$. We also discuss possible improvement of the result.

Subject Index A40, A52

1. Introduction

It is generally a nontrivial and difficult challenge to deduce from a microscopic theory hydrodynamic behavior at long timescales. As was clearly stated by Kadanoff and Martin [1], this is because “*hydrodynamic equations only appear when the behavior is dominated by the secular effects of collisions*” and thus “*most straightforward techniques for determining the correlation functions cannot be successfully applied*”. In the present paper, we will demonstrate in a concrete model that the two-particle irreducible (2PI) framework, which appropriately deals with secular effects through resummation, is indeed able to describe the hydrodynamic behavior from the microscopic level, in particular, a dissipative mode in a two-point correlation function.

There are three different levels of descriptions for the long-time (and long-distance) behavior of a locally equilibrated system. At timescales much longer than the correlation time (the typical timescale of the system), hydrodynamics is established as the framework for the time evolution of conserved densities through macroscopic variables such as temperature, chemical potential, and flow velocity. Diffusion and viscosity constants are regarded as low-energy effective constants. When we study, e.g., the temperature or momentum dependence of the low-energy constants, we need to

include fluctuations around hydrodynamic behavior by employing a “mesoscopic” effective theory. As a typical example, the mode-coupling theory (MCT) [2–5] separates the slow and fast degrees of freedom and describes the effective dynamics of the slow modes while treating the effects of the fast modes as stochastic noise and bare parameter constants. In particular, non-conserved variables such as an order-parameter field φ may be included in addition to hydrodynamic modes in MCT when they contribute to slow dynamics of interest. Only after the microscopic theory is solved will all the phenomenological parameters be fixed, which is the third-level description. However, such a microscopic approach, if applied naively, is suitable only for short-time description; it starts to lose its accuracy at timescales longer than the typical scattering time.

It should also be noticed that hydrodynamics applies at a scale much longer than the correlation length ξ and therefore breaks down as the system approaches a critical point, where $\xi \rightarrow \infty$. MCT was originally devised to describe the slow dynamics near the critical point by including the order-parameter fluctuations φ as well as those of conserved densities, and it is successful in reproducing the singularity of the transport coefficients as the critical point is approached. Accordingly, MCT gives empirical values for the dynamic critical exponent z , which characterizes the dissipation time of a disturbance as $\tau \sim \xi^z$. Such a process should be associated with a dissipative mode whose dispersion is given by $p_0 \propto i|p|^z$, with p_0 and p the mode energy and momentum.

Considering that MCT is just a phenomenological theory and that the dissipative mode (at the critical point) should couple in the two-point function of the order-parameter field φ , it is preferable if we can investigate the low-frequency behavior of the two-point function at the microscopic level. As far as we know, the 2PI approach [6–8] is the only framework that allows us to describe the two-point function at the microscopic level, making the secular effects from multiple scattering of particles tractable through resummation. This motivated us to work on the two-point function of φ in the 2PI framework.

Recently, the 2PI effective action has been actively applied to non-equilibrium phenomena. One of the merits of the 2PI framework is that it treats a condensate $\phi = \langle \varphi \rangle$ and the two-point correlation function of fluctuations $G = \langle (\varphi - \phi)(\varphi - \phi) \rangle$ in a self-consistent way, where $\langle \cdot \rangle$ indicates averaging with a density matrix, and that we can systematically resum higher-order diagrams for ϕ and G while preserving the conservation laws of the system [9–11]. Therefore, the 2PI framework is expected to be very useful in describing the long-time evolution, where resummation of secular effects is crucial.

In the present paper, we analyze the two-point function G of a microscopic theory non-perturbatively, and show how to identify the dissipative mode, which will play an important role in the long-time behavior of the system. Here we take a relativistic $O(N)$ φ^4 scalar theory as a microscopic theory. Another motivation for our study is to see how a dissipative mode appears in the microscopic two-point function of a *relativistic* field theory. The bare mode with the dispersion relation, $p_0^2 = p^2 + m^2$, receives corrections by the interactions in perturbation theory, but is still propagating. On the other hand, we are interested here in the motion in the long-time limit, which corresponds to $p_0 \sim 0$. It has not been fully revealed how dissipative modes emerge microscopically in relativistic field theories. We show that purely dissipative motion is possible in relativistic theory when we include multiple interactions among the modes, which can be efficiently taken into account in the 2PI framework.

After confirming the existence of a dissipative mode in the theory, we briefly extend our discussion to a system near a critical point. Indeed, there is a controversial situation concerning the dynamic critical exponent z between *relativistic* and *nonrelativistic* scalar theories. Analysis of the two-point

function in a nonrelativistic $O(N)$ model [12,13] predicts z close to 2, while a relativistic model gives different values for z depending on the calculation methods: Dynamic renormalization group [14,15] gives $z \sim 1$ while numerical simulation [16] gives $z \sim 2$. In fact, as is shown in the present paper, these analyses (except Ref. [16]) of the two-point function of φ give us the exponent z , not of a purely dissipative mode, but of the propagating modes; the latter motion is faster than the former dissipative modes. This mis-identification is the origin of the controversy.¹

Before closing this introduction, let us comment on several physical situations whose hydrodynamics are of interest. First, the hydrodynamic picture is widely applied to describing the long-time and long-distance behavior of various systems, from quark–gluon plasma (QGP) in heavy-ion collisions [24–26] and the Bose–Einstein condensate of a cold atomic gas [27], to astrophysical phenomena such as supernova explosions [28], and even to the universe itself [29]. In particular, the hydrodynamic picture is a very powerful tool in describing the space-time evolution of QGP, which is an expanding local equilibrium matter made of quarks and gluons. It is in general necessary to figure out the hydrodynamic behavior of QGP including diffusion in any circumstance, but the effects of diffusion are expected to become more relevant if the system is near or at the critical point. In fact, a critical point is conjectured in the T – μ phase diagram of quantum chromodynamics (QCD), where transition from the “hadron phase” (at low temperature T and low chemical potential μ) to the QGP phase (at high T) is of the 2nd order, and heavy-ion collision experiments for the critical point survey have already started [30,31]. Locating the critical point on the QCD phase diagram is one of the main objectives in QGP physics now, and to this end we obviously need correct understanding of the dynamics near the critical point. There are indeed several works applying MCT to the critical dynamics of QCD [21–23,32–34]. However, it is still controversial in the sense that the hydrodynamic mode dominates the critical dynamics. Experimentally, it is crucial to identify the Rayleigh peak (heat-diffusion mode) separating from the Brillouin peaks (propagating sound modes), which is indeed performed in, e.g., inelastic photon scattering by liquid argon and neon [35]. Similar identification should be possible even in QCD matter around the critical point [36]. Second, note also that researchers are trying to describe non-equilibrium phenomena from microscopic theories by exploiting the 2PI framework. For example, the problem of how QGP can be formed within a short time after heavy-ion collisions [37–40] and also non-equilibrium phenomena in cold atomic systems [41] is being discussed. We hope that fundamental understanding on the dissipative modes in the present paper would be helpful for various phenomena, including QGP physics.

This paper is organized as follows: In Sect. 2, we discuss how a dissipative mode emerges from a two-point correlation function using both one-particle irreducible (1PI) and 2PI effective actions. In particular, we focus on the way in which the effects of multiple scattering with a heat bath are included, and propose a criterion for the emergence of a dissipative mode. In Sect. 3, we check the criterion in an $O(N)$ -symmetric φ^4 model within two different calculation methods: 1PI and 2PI frameworks in a $1/N$ expansion at the next-leading order (NLO). We will find that the two-point function in the 1PI–NLO calculation fails to describe the dissipative mode, but that the 2PI–NLO

¹ There are previous studies within MCT treating dynamic critical exponents of nonrelativistic [5,17–20] and relativistic [21–23] field theories. These works, except for Ref. [23], assumed the existence of dissipative modes ($p_0 \sim -i|\mathbf{p}|^2$) and obtained the result $z \sim 2$ at the leading order of $1/N$ or ϵ expansion. Reference [23] obtained a different result, $z \sim 1$, for a relativistic model, claiming that as slow modes one should include a “momentum” variable canonically conjugate to a field variable in relativistic models. We will not address this problem since our scope in this paper is to find a microscopic description of dissipative modes.

calculation can have a dissipative mode with a finite relaxation constant. In Sect. 4, assuming that the dissipative two-point correlation function obtained in the previous section is not modified at the critical point, we estimate the dynamic critical exponent z . We also discuss possible improvement of our result. The last section is devoted to a summary and a discussion.

2. Dissipative mode in the two-point correlation function

We consider a system slightly disturbed from equilibrium and how it undergoes relaxation at timescales much longer than the typical scattering time, by inspecting the two-point function.² Before proceeding to a concrete model analysis, let us first clarify the general properties of two-point functions in an infrared region.

The slowest motion of the system is governed by the hydrodynamic modes consisting of conserved density fluctuations as well as order-parameter fluctuations in the case of critical phenomena, and the Nambu–Goldstone modes in the case of the symmetry-broken phase. At a “mesoscopic” timescale, MCT describes the relaxation processes phenomenologically as an effective theory that involves other slow degrees of freedom in addition to the hydrodynamic modes. Nonlinear couplings among those modes renormalize, e.g., transport coefficients in hydrodynamics.

In the present paper we deal with a non-conserved “order-parameter” field φ in the symmetric phase, for a particular example. According to the phenomenological MCT, the bare two-point correlation function of the field φ with a mass m is given as [5,19,20,44]³

$$G_{\text{pheno}}(p_0, \mathbf{p}) = \frac{1}{-i\Gamma^{-1}p_0 + \mathbf{p}^2 + m^2}. \quad (1)$$

Here Γ is the relaxation constant, which is easily understood after the inverse Fourier transform with respect to t : $G_{\text{pheno}}(t, \mathbf{p}) \sim e^{-\Gamma \cdot (\mathbf{p}^2 + m^2)t}$. On the other hand, in a microscopic field theory, a retarded two-point function $G_R(p_0, \mathbf{p})$ is generally written as

$$G_R(p_0, \mathbf{p}) = \frac{1}{-p_0^2 + \mathbf{p}^2 + m_0^2 - \Sigma_R(p_0, \mathbf{p})}, \quad (2)$$

where $\Sigma_R(p_0, \mathbf{p})$ is a retarded self-energy and m_0 is a bare mass. Henceforth, we indicate retarded and advanced quantities with indices R and A, respectively. The retarded function, G_R , describes the evolution of a single mode excited from the system in thermal equilibrium, and the mode attenuates by scatterings with other modes that play the role of a heat bath. In the present and following sections, we deal with the system off the critical point ($T \neq T_c$) and extract the dissipative term, $i\Gamma^{-1}p_0$, from the self-energy. The system at the critical point will be considered in Sect. 4. Firstly, at $p_0 = 0$, the self-energy $\Sigma_R(p)$ has neither a pole, an essential singularity point, nor a branch point, except when $p_0 = 0$ is approached along the line $|p_0| = |\mathbf{p}|$ with $|\mathbf{p}| \rightarrow 0$ [45,46]. Secondly, although a cut lies on the axis $\Im m p_0 = 0$, the retarded self-energy is defined infinitesimally above this cut in the complex p_0 plane, and, therefore, Σ_R can be expanded around $p_0 = 0$ as long as one keeps $|p_0|/|\mathbf{p}| \ll 1$.⁴

² When a system is in the vicinity of equilibrium, relaxation is described by the retarded two-point function evaluated in equilibrium (linear response theory) [42,43].

³ We can obtain the two-point function for a *conserved* field by replacing Γ with $g\mathbf{p}^2$ in Eq. (1), $C_{\text{pheno}}(p_0, \mathbf{p}) = 1/[-ip_0/(g\mathbf{p}^2) + \mathbf{p}^2 + m^2]$, where g is the transport coefficient related to the conserved quantities.

⁴ Expansions around $p_0 = 0$ are not always justified. For instance, it is known that two-point functions have essential singularities at $p_0 = |\mathbf{p}|$ [45,46], and thus cannot be written as the power series of p_0 around this point.

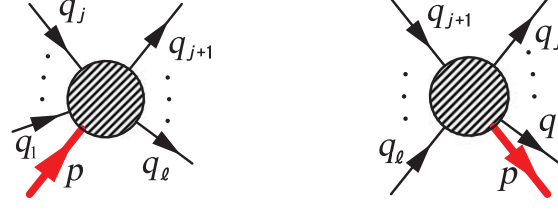


Fig. 1. Scattering processes obtained by cutting ℓ internal lines in a self-energy diagram. The left panel shows the scattering of the particle with momentum \mathbf{p} with j incoming particles and $(\ell - j)$ outgoing particles, while the right panel shows its reverse process.

(This can be explicitly verified using the sunset self-energy diagrams calculated in Ref. [47].) Since the real and imaginary parts of Σ_R are respectively even and odd functions of p_0 [43,48,49], they can be expanded as

$$\begin{aligned}\Re \Sigma_R(p_0, \mathbf{p}) &= \left(c_0 + c_2 \mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4) \right) + \mathcal{O}(p_0^2), \\ \Im \Sigma_R(p_0, \mathbf{p}) &= \left(c_1 + \mathcal{O}(\mathbf{p}^2) \right) p_0 + \mathcal{O}(p_0^3), \quad (T \neq T_c).\end{aligned}\quad (3)$$

Here it should be understood that the dimensions in $\mathcal{O}(p_0^n)$ and $\mathcal{O}(\mathbf{p}^n)$ are canceled with other dimensionful quantities such as temperature T and/or m_0 (and coupling constant λ in general), i.e., $\mathcal{O}((p_0/T)^n), \dots$, etc. When we refer to an “infrared region”, it is always meant that the energy and momentum are much less than these dimensionful parameters. Note that the coefficients c_0 , c_1 , and c_2 depend on λ and T .⁵ With the expression Eq. (3) valid in the infrared region, the two-point function of the microscopic theory can be approximated as

$$G_R(p_0, \mathbf{p}) \sim \frac{1}{-ic_1 p_0 + (1 - c_2) \mathbf{p}^2 + m_0^2 - c_0}. \quad (4)$$

From Eq. (4) we see that the two-point function $G_R(p_0, \mathbf{p})$ of a microscopic theory develops a dissipative mode if the imaginary part of a self-energy is nonzero finite in the infrared limit:

$$c_1 = \lim_{\mathbf{p} \rightarrow 0} \lim_{p_0 \rightarrow 0} \left(\frac{\partial}{\partial p_0} \Im \Sigma_R(p_0, \mathbf{p}) \right) \neq 0, \quad \infty. \quad (5)$$

We propose to check the finiteness of this quantity as a criterion for the existence of a dissipative mode, and to identify c_1 as the inverse of the phenomenological relaxation constant Γ^{-1} in Eq. (1). Later we will show that it is indeed nonzero finite at the NLO in the $1/N$ expansion in the 2PI formalism.

According to the Cutkosky rules generalized at finite temperature [43,50–53], one can relate $\Im \Sigma_R$ to the sum of cut diagrams. The cut diagrams, in which ℓ internal lines are cut, can be interpreted as the scattering processes shown in Fig. 1, where an incoming (outgoing) particle with $p = (p_0, \mathbf{p})$ disappears (emerges) through the scattering with particles in the heat bath. More precisely, shown in the left diagram of Fig. 1 is the process where a particle with momentum p scatters off j particles with momenta q_1, \dots, q_j in the heat bath, and turns to $\ell - j$ particles with momenta q_{j+1}, \dots, q_ℓ in the final state. The right diagram shows the reverse process.

⁵ Therefore, $1 - c_2$ in Eq. (4) does not vanish identically. As we will see, coefficients c_0 and (c_1, c_2) depend on the coupling constant λ and are of the order of N^0 and N^{-1} , respectively, in the $O(N)$ scalar model.

Consider the soft limit of interest $p \rightarrow 0$, where the energy–momentum conservation between the initial and final states, $q_1 + \cdots + q_j = q_{j+1} + \cdots + q_\ell$, must be fulfilled. Clearly, the process in which there are no incoming particles from the heat bath (i.e., the initial total energy–momentum is zero) does not contribute to Eq. (5) because the condition for the final total energy–momentum $q_1 + \cdots + q_\ell = 0$ can never be satisfied for massive particles in the heat bath. From this simple observation, one concludes that $\Im \Sigma_R(p_0 \rightarrow 0, \mathbf{p} \rightarrow 0) = 0$ at $T = 0$, i.e., there is no infrared dissipation in the vacuum. At $T \neq 0$, however, the imaginary part of the self-energy $\Im \Sigma_R(p_0 \rightarrow 0, \mathbf{p} \rightarrow 0)$ receives, in general, contributions from the processes where the incoming particle scatters off particles in the heat bath and contributions from their reverse processes.

A more careful inspection reveals that the imaginary part of the self-energy $\Im \Sigma_R(p_0, \mathbf{p})$ can be written in a rather intuitive form:

$$\begin{aligned} \Im \Sigma_R(p_0, \mathbf{p}) &\sim \sum_{\ell} \sum_{j=0}^{\ell} \int dq_1 \cdots dq_{\ell} \rho^{(j)}(p; q_1, \dots, q_j) \rho^{(\ell-j)}(q_{j+1}, \dots, q_{\ell}) \\ &\times \{ \mathcal{N}[f_{q_1}, \dots, f_{q_{\ell}}] - \tilde{\mathcal{N}}[f_{q_1}, \dots, f_{q_{\ell}}] \}, \end{aligned} \quad (6)$$

where $\rho^{(j)}$ and $\rho^{(\ell-j)}$ are, respectively, spectral functions for incoming and outgoing thermal particles, and $\mathcal{N}[\{f_i\}]$ is the statistical weight of particles participating in this process ($\tilde{\mathcal{N}}[\{f_i\}]$ is for the reverse process). The weight $\mathcal{N}[\{f_i\}]$ and $\tilde{\mathcal{N}}[\{f_i\}]$ will be expressed as a product of thermal distribution functions $f_p = (e^{\beta|p_0|} - 1)^{-1}$ as follows:

$$\begin{aligned} \mathcal{N}[\{f_i\}] &= [f_{q_1} \cdots f_{q_j}] \times [(1 + f_{q_{j+1}}) \cdots (1 + f_{q_{\ell}})], \\ \tilde{\mathcal{N}}[\{f_i\}] &= [(1 + f_{q_1}) \cdots (1 + f_{q_j})] \times [f_{q_{j+1}} \cdots f_{q_{\ell}}]. \end{aligned} \quad (7)$$

One finds that the product of f_q denotes incoming particles from the heat bath, and that of $(1 + f_q)$ denotes outgoing particles into the heat bath. The spectral functions, $\rho^{(j)}$ and $\rho^{(\ell-j)}$, specify kinematical windows for scattering states. Therefore, the quantity $\Im \Sigma_R(p_0, \mathbf{p})$ is nonvanishing when the kinematical windows for the incoming and outgoing particles overlap with each other. This is a very useful point of view when we evaluate $\Im \Sigma_R(p_0, \mathbf{p})$ in a concrete model. We will indeed see below that $\Im \Sigma_R(p_0, \mathbf{p})$ in the $O(N)$ scalar model can be written in this form, and we can assess if it is zero or nonzero by checking the kinematical windows.

Note that we will also treat “composite” fields such as φ^2 , since the Schwinger–Dyson equation relates two-point functions to higher-point functions, which could have two (or more) fields at the same space-time point. A similar consideration to that discussed above should hold for those “composite” fields.

3. Relaxation constant at NLO in the $1/N$ expansion

In this paper, as a typical example of relativistic quantum field theories, we employ an $O(N)$ -symmetric φ^4 model in $(d + 1)$ -dimensional space-time with the action

$$S[\varphi] = \int dx_0 d\mathbf{x} \left[\frac{1}{2} \partial_\mu \varphi_a(x) \partial^\mu \varphi_a(x) - \frac{m_0^2}{2} \varphi_a(x) \varphi_a(x) - \frac{\lambda_0}{4!N} (\varphi_a(x) \varphi_a(x))^2 \right], \quad (8)$$

where $\varphi_a(x) = \varphi_a(x_0, \mathbf{x})$ ($a = 1, \dots, N$) is an N -component real scalar field, m_0^2 and λ_0 are the bare mass and coupling constant, and we assume that $d \leq 3$ so that the action is renormalizable.

The partition function in the imaginary-time formalism is given by

$$Z[J, K] = \int_{\varphi(0, \mathbf{x}) = \varphi(-i\beta, \mathbf{x})} \mathcal{D}\varphi \exp \left[-S[\varphi] + \int dx_0 d\mathbf{x} J_a(x) \varphi_a(x) + \frac{1}{2} \int dx_0 d\mathbf{x} dy_0 d\mathbf{y} \varphi_a(x) K_{ab}(x, y) \varphi_b(y) \right], \quad (9)$$

where β is an inverse temperature $\beta = 1/T$, and we have introduced one- and two-point source fields, J and K , respectively (for reviews of field theories at finite temperature, see Refs. [43,48,49]). For instance, the one- and two-point functions are obtained from the functional derivative of $Z[J, K]$,

$$\begin{aligned} \langle \varphi_a(x) \rangle &\equiv \text{Tr}[\rho_{\text{eq}} \varphi_a(x)] \\ &= \frac{\delta}{\delta J_a(x)} \ln Z[J, K] \Big|_{J=K=0}, \\ \langle \delta \varphi_a(x) \delta \varphi_b(y) \rangle &\equiv \text{Tr}[\rho_{\text{eq}} \delta \varphi_a(x) \delta \varphi_b(y)] \\ &= \frac{\delta^2}{\delta J_a(x) \delta J_b(y)} \ln Z[J, K] \Big|_{J=K=0} \quad (x_0 > y_0) \\ &= 2 \frac{\delta}{\delta K_{ab}(x, y)} \ln Z[J, K] \Big|_{J=K=0} - \langle \varphi_a(x) \rangle \langle \varphi_b(y) \rangle \quad (x_0 > y_0), \end{aligned}$$

where $\delta \varphi = \varphi - \langle \varphi \rangle$ and ρ_{eq} denotes the density matrix of the canonical ensemble, $\rho_{\text{eq}} = e^{-\beta H}$. From this partition function $Z[J, K]$, one can introduce the 2PI effective action via a double Legendre transform in J and K , while the ordinary 1PI effective action is defined as the single Legendre transform in J with $K \equiv 0$. The advantage of the 2PI effective action is that it reorganizes the perturbative expansion series by resumming the higher-order terms self-consistently into the two-point correlation function. Details of the formalism of the 2PI effective action are presented in Ref. [8].

Because dissipation will result from multiple scatterings of the mode in the heat bath, one needs some sort of non-perturbative framework that can treat the processes involving infinitely many interactions. Thus, in the following calculation, we employ the $1/N$ expansion, which is a typical non-perturbative expansion. The leading-order (LO) contribution is the so-called tadpole diagram, which only brings in a shift of the mass $m_0^2 \rightarrow m^2$. Here we restrict ourselves to the symmetric phase: $m^2 > 0$. In the symmetric phase, the condensate of the order parameter is zero: $\langle \varphi \rangle = 0$. Henceforth, we write $\delta \varphi$ as φ for brevity. The scattering processes among the particles appear at the NLO and in higher-order diagrams. The NLO diagram for the self-energy in the 2PI formalism that is relevant for the scattering processes in the symmetric phase⁶ is found to be

$$\begin{aligned} \Sigma^{\text{scatt.}}(i\omega_n, \mathbf{p}) &= \text{Diagram with a horizontal line, an incoming arrow labeled } i\omega_n, \mathbf{p} \text{ from the left, and an outgoing arrow labeled } i\omega_{n+m}, \mathbf{p}+\mathbf{q} \text{ to the right. A wavy loop connects the two vertices.} \\ &= \text{Diagram 1} + \text{Diagram 2} + \dots + \text{Diagram N}, \quad (10) \end{aligned}$$

⁶ Full diagrams contributing to a 2PI self-energy at NLO in both symmetric ($\langle \varphi \rangle = 0$) and broken ($\langle \varphi \rangle \neq 0$) phases are given in Ref. [54].

In particular, in the LO approximation for D_R ,

$$\rho_D(p_0, \mathbf{p}) = \frac{2 \Im \Pi_R(p_0, \mathbf{p})}{\left[1 + \frac{\lambda}{6} \Re \Pi_R(p)\right]^2 + \left[\frac{\lambda}{6} \Im \Pi_R(p_0, \mathbf{p})\right]^2}. \quad (19)$$

Note that the expression Eq. (17) for $\Im \Sigma_R$ at NLO is valid in both the 1PI and 2PI formalisms, except that G is the free two-point function in the 1PI formalism while it is the full self-consistent two-point function in the 2PI formalism. Note also that Eq. (17) can be rewritten in the form alluded to earlier in Eq. (6), so that the effects of the heat bath can be explicitly seen:

$$\begin{aligned} & \rho(q) \rho_D(p+q) [n(q_0) - n(p_0 + q_0)] \\ &= \begin{cases} \rho(q) \rho_D(p+q) [f_q (1 + f_{p+q}) - (1 + f_q) f_{p+q}] & (p_0 + q_0 \geq 0, q_0 \geq 0), \\ \rho(q) \rho_D(p+q) [f_q f_{p+q} - (1 + f_q) (1 + f_{p+q})] & (p_0 + q_0 \geq 0, q_0 < 0), \end{cases} \end{aligned}$$

and two other combinations for p_0 and q_0 should be obvious. Now one can judge the finiteness of the relaxation constant through the inspection of overlapping kinematical windows specified by the spectral functions ρ and ρ_D .

In the infrared region, $\Im \Sigma_R(p)$ can be approximated as

$$\Im \Sigma_R(p) = -\frac{\lambda}{6N} \frac{\lambda}{6} \int \frac{dq_0 d\mathbf{q}}{(2\pi)^{d+1}} \rho(q_0, \mathbf{q}) \rho_D(q_0, \mathbf{q}) p_0 n'(q_0) + \mathcal{O}(\mathbf{p}^2, p_0^2). \quad (20)$$

Substituting Eq. (20) into Eq. (5), we find the equation that determines c_1 , which we now identify as the inverse of the phenomenological relaxation constant Γ^{-1} . The aim of the present paper is to show that Γ is nonzero finite. For notational brevity, we define a constant γ by

$$\frac{1}{\gamma} \equiv - \int \frac{dq_0 d\mathbf{q}}{(2\pi)^{d+1}} \rho(q_0, \mathbf{q}) \rho_D(q_0, \mathbf{q}) n'(q_0), \quad (21)$$

which is proportional to Γ . In the following subsections, we will examine if γ is nonzero finite at the NLO of the $1/N$ expansion in the 1PI and 2PI formalisms.⁸

3.1. 1PI–NLO evaluation

First, we confirm that a dissipative mode does *not* emerge in the 1PI two-point function at the NLO. In the $1/N$ expansion in the 1PI formalism, the two-point function G_R appearing in the diagrams is the free two-point function obtained at LO with the tadpole effect included in the self-energy, whose spectral function is

$$\rho(p_0, \mathbf{p}) = 2\pi \operatorname{sign}(p_0) \delta(p_0^2 - \mathbf{p}^2 - m^2). \quad (22)$$

The other function $\rho_D(p_0, \mathbf{p})$ in Eq. (21) is the spectral function for a “composite” φ^2 field. Because the $\lambda\varphi^4$ interaction couples the φ^2 field with two φ fields in either way of $\varphi^2\varphi \rightarrow \varphi$ or $\varphi^2 \rightarrow \varphi\varphi$

⁸ One may wonder if the relaxation constant Γ or equivalently γ defined in Eq. (21) is alternatively defined in terms of the current commutators through the Kubo formula, just like transport coefficients. Recall, however, that the Kubo formula evaluates the transport of conserved densities. On the other hand, the quantity $\langle\varphi\rangle$ here is non-conserved and can relax locally. Therefore, the relaxation constant Γ is naturally given by Eq. (5) or Eq. (21).

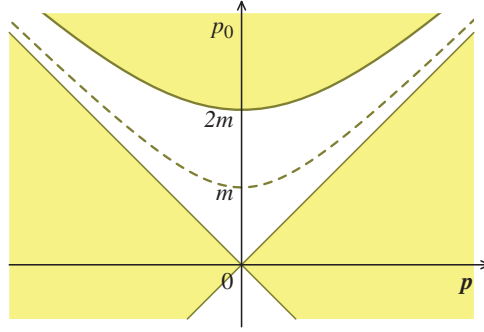


Fig. 2. The region on the p_0 - \mathbf{p} plane where $\rho_D(p_0, \mathbf{p})$ has a nonzero value in the $1/N$ expansion at NLO in the 1PI framework: one is a space-like region $s = p_0^2 - \mathbf{p}^2 < 0$, and the other is a two-particle continuum region $s > (2m)^2$.

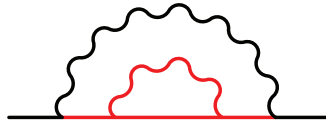


Fig. 3. One of the NNLO diagrams in the 1PI calculation. The sub-diagram in red (the inside loop diagram) can be interpreted as a dressing of the two-point function G , which gives nonzero spectral strength except for $p_0 = 0$ on the p_0 - \mathbf{p} plane.

(or their reverse processes), the spectral function $\rho_D(p_0, \mathbf{p})$ is nonzero in two kinematical regions: the space-like region $s = p_0^2 - \mathbf{p}^2 < 0$ and two-particle continuum region $s > (2m)^2$ (Fig. 2).

The supports of the spectral functions $\rho(p_0, \mathbf{p})$ and $\rho_D(p_0, \mathbf{p})$ do not overlap, and therefore the integral Eq. (21) for $(1/\gamma)_{\text{1PI}}$ at NLO vanishes. Therefore, we conclude that, in the 1PI–NLO calculation, dissipation is not seen in the infrared limit of the φ mode.

One obvious way to proceed is to go beyond the NLO in the 1PI framework. At the next-to-next-leading order (NNLO), the expression for the relaxation constant becomes more involved than Eq. (21), but it certainly gives a nonzero value because it is known that the spectral functions ρ and ρ_D become nonzero except for $p_0 = 0$ on the p_0 - \mathbf{p} plane once the so-called sunset diagram is included in the self-energy [47,49,57]. The sunset diagram appears as a sub-diagram in the NNLO–1PI calculation (see, e.g., Fig. 3, where the wavy line in red includes a sunset diagram). We will report progress on this approach in a separate paper.

Notice that the sunset diagram is already resummed at NLO in the 2PI effective action. Thus, in this paper, we take another direction. While keeping the NLO approximation, we exploit the 2PI effective action formalism, which self-consistently takes into account the processes relevant to dissipation in the self-energy of the two-point function G . Indeed, inclusion of the sunset diagram alone is known to give a finite Γ [58,59], although it is not the full NLO calculation. Below, we develop a formalism that allows us to treat all the NLO diagrams including the sunset diagram.

3.2. 2PI–NLO evaluation

Let us now confirm that the existence of a dissipative mode is accommodated in the 2PI–NLO calculation by showing that Eq. (21) is nonzero finite. As we have already emphasized several times, the 2PI effective action self-consistently determines two-point functions (and the condensate $\langle \varphi \rangle$ if $O(N)$ symmetry is broken). The self-consistent structure is seen in the expression Eq. (17) of $\Im m \Sigma_R(p_0, \mathbf{p})$ at NLO: It is given by the spectral functions ρ and ρ_D , but they are in turn given as

imaginary parts of two-point functions. Therefore, the precise form of $\Im \Sigma_R(p_0, \mathbf{p})$ is found only after the Schwinger–Dyson-type equation is solved self-consistently. We are going to show, however, from general properties of the spectral functions ρ and ρ_D that $\Im \Sigma_R(p_0, \mathbf{p})$ in 2PI–NLO has a term proportional to p_0 with a finite coefficient.

Recall that, in the 1PI–NLO calculation, the supports of the two spectral functions ρ and ρ_D do not overlap in the (q_0, \mathbf{q}) space, giving a vanishing result for Eq. (21), but that 1PI–NNLO will give a nonzero finite result thanks to the sunset diagram as included in Fig. 3. The same contribution is included in the 2PI–NLO diagrams (written in terms of full two-point functions) and thus Eq. (21) is expected to become nonzero finite.

We will see this property below. Because the spectral functions are odd in the frequency q_0 and $n'(x) = n'(-x) = -e^x/(e^x - 1)^2$, Eq. (21) yields

$$\frac{1}{\gamma} = \frac{\Omega_d}{(2\pi)^{d+1}} \int_0^\infty dq_0 \int_0^\infty d|\mathbf{q}| |\mathbf{q}|^{d-1} \rho(q_0, |\mathbf{q}|) \rho_D(q_0, |\mathbf{q}|) \frac{e^{\beta q_0}}{(e^{\beta q_0} - 1)^2}, \quad (23)$$

where $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the d -dimensional sphere.⁹ Since the spectral functions ρ and ρ_D are positive semi-definite for $q_0 > 0$ in general, and, as mentioned above, they have nonzero values except for $q_0 = 0$ in the integration domain in the 2PI–NLO calculation [47,49,57], we conclude at this point that Eq. (23) has a nonzero value, in contrast to the 1PI–NLO result.

In the following, we verify that Eq. (23) for $1/\gamma$ is not diverging, but finite. To this end, we first notice that the spectral functions would not have singularities. For example, the delta-function singularity of ρ coming from the one-particle pole in the vacuum turns into a peak with a finite width due to the interactions in the heat bath via resummation in the 2PI formalism. Therefore, we can reasonably assume that the integrand does not have any singularity.

Then, what remains to be checked is the behavior of the integrand in the infrared (IR) and ultra-violet (UV) regions. For that purpose, we change the variables from $(q_0, |\mathbf{q}|)$ to (Q, θ) , defined by $q_0 = Q \cos \theta$, $|\mathbf{q}| = Q \sin \theta$:

$$\frac{1}{\gamma} = \frac{\Omega_d}{(2\pi)^{d+1}} \int_0^{\pi/2} d\theta \int_0^\infty Q dQ (Q \sin \theta)^{d-1} \rho(Q, \theta) \rho_D(Q, \theta) \frac{e^{\beta Q \cos \theta}}{(e^{\beta Q \cos \theta} - 1)^2}. \quad (24)$$

In terms of these new variables, the IR and UV limits correspond to $Q \rightarrow 0$ and $Q \rightarrow \infty$, respectively, while there is no restriction on the angle θ . Thus, we evaluate the integrand in these limiting regions, and then perform the integration over θ to check the finiteness of the integral.

Consider the IR region, $\beta Q \ll 1$, where the integrand of Eq. (24) is approximated as

$$Q^d (\sin \theta)^{d-1} \frac{\rho(Q, \theta)}{Q \cos \theta} \frac{\rho_D(Q, \theta)}{Q \cos \theta}. \quad (25)$$

In the IR limit $Q \rightarrow 0$, the spectral functions ρ and ρ_D become zero as long as the mass is kept finite.¹⁰ Thus, the integrand, Eq. (25), is bounded by Q^a ($a < d - 2$) in the IR region and the

⁹ The spectral functions are assumed to be rotationally invariant in equilibrium without external fields.

¹⁰ However, we expect that the spectral function ρ will diverge in the IR limit if one considers the massless limit. As we will comment later, such a divergent behavior will affect the critical dynamics of the two-point functions. Estimation of this divergence is now under investigation.

Q -integral is IR-finite, when the spatial dimension d satisfies $d \geq 1$. Then, we consider the θ -integral. A divergence may arise in the vicinity of $\theta = \pi/2$. In this case, $q_0/|q| \ll 1$ is satisfied, and, therefore, $\rho = 2 \Im m G_R$ and $\rho_D = 2 \Im m D_R$ are proportional to the frequency $q_0 = Q \cos \theta$ (see the argument above Eq. (3)). We find that the θ -integral is also finite for $d > 0$. Hence, we conclude that no divergence appears from the IR region at least for $d \geq 1$, which includes the dimensions of interest, $1 \leq d \leq 3$.

Next, we do not expect any divergence either in the UV limit $Q \rightarrow \infty$, because physically nonzero $1/\gamma$ should be a purely in-medium effect, as is seen in the integrand that contains $n'(q_0)$. As long as $\theta \neq \pi/2$, the last factor in the integrand gives an exponential suppression $\sim e^{-\beta Q \cos \theta}$ as $Q \rightarrow \infty$. The remaining part gives at most a positive power of Q , as we will see later. Therefore, there is no UV divergence for $\theta \neq \pi/2$.

The only concern is the integrand when the angle θ is close to $\pi/2$, namely, in a deep space-like region $q_0 = Q \cos \theta \rightarrow 0$, $q = Q \sin \theta \rightarrow \infty$. No matter how large Q is, we can take the quantity $\beta Q \cos \theta$ small enough by choosing θ very close to $\pi/2$. Then, we find a different estimate $\sim 1/(\beta Q \cos \theta)^2$ for the last factor in the integrand. Let us define an angle α that satisfies $\beta Q \cos \alpha \ll 1$, or, equivalently, $\pi/2 - \alpha \ll \frac{1}{\beta Q}$. We estimate the UV behavior of Eq. (24) by putting α as the lower limit of the θ -integral (we suppress the lower limit of the Q -integral here because we are just studying the UV behavior of the integrand):

$$\begin{aligned} & \frac{\Omega_d}{(2\pi)^{d+1}} \int_{\alpha}^{\pi/2} d\theta \int^{\infty} Q dQ (Q \sin \theta)^{d-1} \rho(Q, \theta) \rho_D(Q, \theta) \frac{e^{\beta Q \cos \theta}}{(e^{\beta Q \cos \theta} - 1)^2} \\ & \leq \frac{\Omega_d}{(2\pi)^{d+1}} \int_{\alpha}^{\pi/2} d\theta \int^{\infty} Q dQ (Q \sin \theta)^{d-1} \rho(Q, \theta) \rho_D(Q, \theta) \frac{1}{(\beta Q \cos \theta)^2}. \end{aligned} \quad (26)$$

Since ρ and ρ_D are proportional to $q_0 = Q \cos \theta$ when βq_0 is small enough, the integrand is regular in the limit $\beta Q \cos \theta \rightarrow 0$. Let us now introduce two functions as follows:

$$h_1(Q) \equiv \lim_{\theta \rightarrow \pi/2} \frac{\rho(Q, \theta)}{Q \cos \theta}, \quad h_2(Q) \equiv \lim_{\theta \rightarrow \pi/2} \frac{\rho_D(Q, \theta)}{Q \cos \theta}. \quad (27)$$

Then, the θ -dependence of the integrand can be neglected, and the right-hand side of Eq. (26) is evaluated as

$$\sim \frac{\Omega_d}{(2\pi)^{d+1}} \frac{1}{\beta^2} \int^{\infty} \left(\frac{\pi}{2} - \alpha \right) Q dQ Q^{d-1} h_1(Q) h_2(Q), \quad (28)$$

where we have used the approximations $\sin \theta \sim 1$ and $\cos \alpha \sim \pi/2 - \alpha$, valid in the vicinity of $\theta = \pi/2$. Using the condition $\pi/2 - \alpha \ll \frac{1}{\beta Q}$, we can further put an upper limit on the integration shown in Eq. (28):

$$\frac{\Omega_d}{(2\pi)^{d+1}} \frac{1}{\beta^2} \int^{\infty} \left(\frac{\pi}{2} - \alpha \right) dQ Q^d h_1(Q) h_2(Q) \ll \frac{\Omega_d}{(2\pi)^{d+1}} \frac{1}{\beta^3} \int^{\infty} dQ Q^{d-1} h_1(Q) h_2(Q). \quad (29)$$

Now the problem has reduced to checking the UV behavior of the quantity:

$$Q^{d-1} h_1(Q) h_2(Q). \quad (30)$$

If this product decays faster than $1/Q$ as $Q \rightarrow \infty$, the integral is UV-finite. In order to identify the high-momentum behavior of the functions h_1 and h_2 , a naive dimensional analysis is sufficient.

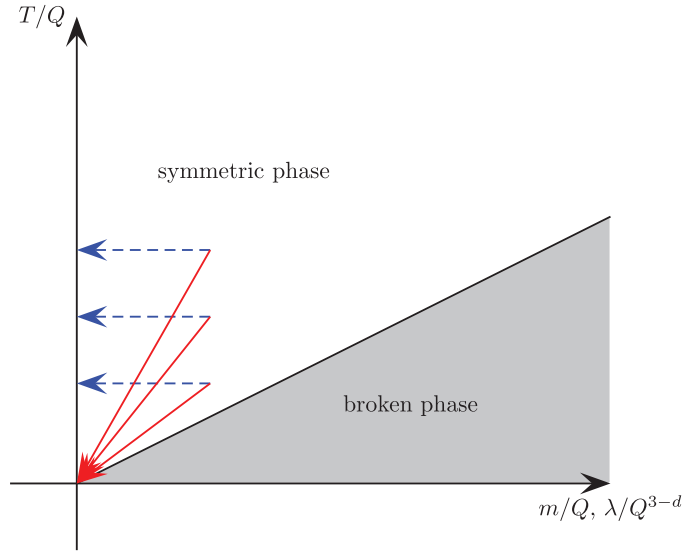


Fig. 4. A schematic phase diagram in the 3D space $(T/Q, m/Q, \lambda/Q^{3-d})$. The surface defined by $T = T_c$ separates the broken phase (shaded) from the symmetric one (unshaded). The solid arrows indicate some limits of $Q \rightarrow \infty$ with T, m , and λ fixed at certain values or alternatively the limits of $T, m, \lambda \rightarrow 0$ with their ratios $T : m : \lambda^{1/(3-d)}$ being fixed and with Q being finite. The dashed arrows indicate some non-interacting limits, where the $\lambda \rightarrow 0$ limit is taken with T, m , and Q fixed at certain values.

Since the mass dimensions of ρ and ρ_D are -2 and $d-3$, respectively, those of h_1 and h_2 in Eq. (27) become -3 and $d-4$. We define two dimensionless functions \tilde{h}_1 and \tilde{h}_2 as follows:

$$h_1 = \frac{1}{Q^3} \tilde{h}_1 \left(\frac{T}{Q}, \frac{m}{Q}, \frac{\lambda}{Q^{3-d}} \right), \quad h_2 = \frac{1}{Q^{4-d}} \tilde{h}_2 \left(\frac{T}{Q}, \frac{m}{Q}, \frac{\lambda}{Q^{3-d}} \right), \quad (31)$$

where we have provided the overall mass dimensions by Q and explicitly shown the dependence on other dimensionful parameters T, m , and λ . As $Q \rightarrow \infty$, h_1 and h_2 decay at least as fast as $1/Q^3$ and $1/Q^{4-d}$, respectively, if \tilde{h}_1 and \tilde{h}_2 do not diverge in this limit. This is indeed the case as long as we stay in the symmetric phase, as seen from a plausible argument shown below.

Recall that the self-energy given by Eq. (10) is valid in the symmetric phase. Let $T_c(m, \lambda)$ be the critical temperature below which the $O(N)$ symmetry is broken for a given mass m and a given coupling constant λ . Then, $T = T_c(m, \lambda)$ defines a surface in the three dimensional space $(T/Q, m/Q, \lambda/Q^{3-d})$, which separates the broken and symmetric phases as shown in Fig. 4. Let us first consider the limit $\lambda/Q^{3-d} \rightarrow 0$ with fixed Q and with some fixed T and m , as shown by dashed arrows in Fig. 4, before discussing the behavior of \tilde{h}_1, \tilde{h}_2 in the limit of $T/Q, m/Q, \lambda/Q^{3-d} \rightarrow 0$ with fixed ratios of $T : m : \lambda^{1/(3-d)}$. In this non-interacting limit, one can explicitly evaluate the spectral functions ρ and ρ_D . Now ρ is given by the delta-function Eq. (22), so that ρ , and therefore \tilde{h}_1 , does not diverge in the region with which we are concerned, $|q|/|q_0| \gg 1$. Thus, \tilde{h}_1 depends on λ with a positive power in the space-like region. The other spectral function ρ_D is given by a single bubble diagram (see Eqs. (12), (18)):

$$\rho_D(q) = 2\Im m \Pi_R(q). \quad (32)$$

In the limit $\lambda \rightarrow 0$, the two-point function in the diagram can be replaced by a free two-point function:

$$G_R(q) = \frac{1}{-(q_0 + i\epsilon)^2 + \mathbf{q}^2 - m^2}, \quad (33)$$

where ϵ is an infinitesimal positive number. After analytic calculation, we obtain the UV behavior of h_2 as follows:

$$\begin{aligned}
 h_2 &= \lim_{q_0 \rightarrow 0} \frac{\rho_D(q)}{q_0} \\
 &= \frac{\pi \Omega_{d-1}}{(2\pi)^d} \frac{T^{d-3}}{|q|} \int_{\sqrt{|q|^2+4m^2}/2T}^{\infty} du \frac{e^u}{(e^u - 1)^2} \left(u^2 - \frac{m^2}{T^2}\right)^{(d-3)/2} \\
 &\leq \frac{\pi \Omega_{d-1}}{(2\pi)^d} \frac{T^{d-3}}{|q|} \int_{\sqrt{|q|^2+4m^2}/2T}^{\infty} du \frac{e^u}{(e^u - 1)^2} \\
 &= \frac{\pi \Omega_{d-1}}{(2\pi)^d} \frac{1}{|q|^{4-d}} \left(\frac{T}{|q|}\right)^{d-3} \frac{1}{e^{\sqrt{q^2+4m^2}/2T} - 1}. \tag{34}
 \end{aligned}$$

This inequality holds for $d \leq 3$ and $|q|/2T \geq 1$, the latter of which is always satisfied in the UV region. Therefore, h_2 decays at least as fast as $1/|q|^{4-d} \propto 1/Q^{4-d}$, which is consistent with the naive dimensional counting as shown in Eq. (31). It should also be noted that, due to the exponential part in the last line of Eq. (34), h_2 can be bounded by $1/Q^{4-d}$ even if we take the limit $T \rightarrow 0$ or $m \rightarrow 0$.

Now let us come back to the relevant limit $Q \rightarrow \infty$ with T, m , and λ fixed in the symmetric phase, as shown by solid arrows in Fig. 4. In this limit, we approach the origin of the three dimensional space $(T/Q, m/Q, \lambda/Q^{3-d})$. The same limit can be achieved by making $T, m, \lambda \rightarrow 0$ with fixed ratios $T : m : \lambda^{1/(3-d)}$ for a finite Q . One can also approach the origin by taking the non-interacting limit $\lambda \rightarrow 0$ first and then by setting T and m to zero. It is physically quite plausible that the two limits are the same as long as they are taken in the symmetric phase, where no singularity is expected.

In view of the above, we conclude that h_1 and h_2 decay as fast as Q^{-3} and Q^{d-4} , respectively. Then, the dimension of the integrand in Eq. (29) becomes Q^{2d-8} and thus the integral is convergent for $d \leq 3$, which are the dimensions of interest.

Hence, in case of the spatial dimension $1 \leq d \leq 3$, Eq. (24) becomes nonzero and finite constant. Thus, we finally conclude that

$$\left(\frac{1}{\gamma}\right)_{2\text{PI}} \neq 0, \infty. \tag{35}$$

This means that a two-point correlation function obtained from the self-consistent equation in 2PI effective action has a dissipative mode in the IR region.

Some comments are in order. First of all, we emphasize that multi-scattering processes in the heat bath are taken into account in the 2PI–NLO but not in the 1PI–NLO calculation. As a result the dissipative mode, which does not exist in the free theory, appears non-perturbatively in the 2PI–NLO calculation. Second, it should be noticed that the dissipative mode emerges irrespective of whether the microscopic field theory is *relativistic* or *nonrelativistic*. This fact implies the same scaling behavior of the dissipative mode for the *relativistic* and *nonrelativistic* field theory at the critical point. These are in contrast to the propagating modes. The propagating modes exist in the free theory and have different dispersion relations in relativistic ($p_0^2 \sim |\mathbf{p}|^2 + m^2$) and nonrelativistic cases ($p_0 \sim |\mathbf{p}|^2$).¹¹

¹¹ The propagating modes have real frequencies with small decay rates, while the dissipative modes are overdamped.

They get modified by scatterings in the heat bath to acquire damping widths as well as energy shifts. Therefore, the scaling behavior of the propagating modes for the *relativistic* and *nonrelativistic* field theories is in general different. Third, we have not attempted to obtain a numerical value of the relaxation constant, although it is in principle possible if one obtains the two-point functions around thermal equilibrium. Namely, if one solves the Kadanoff–Baym equation near equilibrium, then one can evaluate $1/\gamma$ defined in Eq. (21) with the resultant solutions. We leave this problem for our future work. Rather, in the next section, we discuss an important consequence deduced from the presence of the diffusion term, even without knowing its coefficient.

4. Implications for the dynamic exponent z

So far, we have considered the case where the effective mass $m^2 \equiv m_0^2 - \Re \Sigma_R(p=0)$ is nonzero. Roughly speaking, the limit $m \rightarrow 0$ corresponds to the critical point where the correlation length diverges, $\xi \sim 1/m \rightarrow \infty$. This divergence also affects the time evolution of a disturbed system towards equilibrium. Off the critical point $m \neq 0$, small disturbances die away exponentially with the relaxation time τ . As the critical point is approached, the relaxation time becomes enormously long, which is called critical slowing down, and we define the dynamic critical exponent z via $\tau \sim \xi^z$ [2,3,60–62]. In this section, we first present a consequence on the dynamic critical exponent z from the dissipative mode identified in the previous section, assuming that the relaxation constant Γ remains nonzero finite at the critical point. Then we remark on a possible modification of such a naive assumption.

In order to identify the dynamic critical exponent z , we exploit the dynamic scaling relation at the critical point. We assume that under a scale transformation the two-point function transforms anisotropically with respect to p_0 and \mathbf{p} :

$$G_R(p_0, \mathbf{p}) = b^{2-\eta} G_R(b^z p_0, b\mathbf{p}), \quad (36)$$

where b is a dimensionless scale factor that is taken bigger than one: $b > 1$. The prefactor $b^{2-\eta}$ is fixed by the static part of the two-point function characterized by an exponent η . Indeed, if one substitutes $b = \Lambda/|\mathbf{p}|$ into Eq. (36) and sets $p_0 = 0$, one recovers the static scaling for the correlation function $G_R^{\text{static}}(\mathbf{p}) \sim (\Lambda/|\mathbf{p}|)^{2-\eta}$, where an arbitrary dimensionful parameter Λ is introduced to make the combination $\Lambda/|\mathbf{p}|$ dimensionless. We take Λ of the order of the critical temperature T_c . Next, we will rewrite Eq. (36) into a more useful form to evaluate z . Since the mass dimension of G_R is -2 , we can write $G_R(p_0, \mathbf{p})$ as [5]:

$$G_R(p_0, \mathbf{p}) = \left(\frac{\Lambda}{|\mathbf{p}|}\right)^{2-\eta} \cdot G_R\left(\frac{\Lambda^z p_0}{|\mathbf{p}|^z}, \Lambda\right) = \Lambda^{-2} \left(\frac{\Lambda}{|\mathbf{p}|}\right)^{2-\eta} \cdot \tilde{G}_R\left(\Lambda^{z-1} \frac{p_0}{|\mathbf{p}|^z}\right). \quad (37)$$

Thus, we can extract z from the different scaling properties of p_0 and \mathbf{p} in the two-point function.

The self-consistent equation obtained in the 2PI framework has previously been utilized to evaluate the static critical exponents η [63] and ν [56] (see Refs. [63–65] for application of the 2PI framework to 2nd-order phase transitions). While the imaginary part of the self-energy $\Im \Sigma_R(p)$ vanishes in the static limit, the momentum dependence of the real part $\Re \Sigma_R(p)$ brings in the scaling form of the two-point function:

$$G_R^{\text{static}}(|\mathbf{p}|; T_c) = \Lambda^{-2} \left(\frac{\Lambda}{|\mathbf{p}|}\right)^{2-\eta}. \quad (38)$$

Of course, this is the same form as identified in the prefactor in Eq. (36). Since this scaling function is simply obtained from $\Re \Sigma_R(p)$, we can substitute it into the two-point function Eq. (4) to find the expression at the critical point:

$$G_R(p; T_c) = \frac{1}{-i\Gamma^{-1}p_0 + \Lambda^\eta |\mathbf{p}|^{2-\eta}} = \Lambda^{-2} \left(\frac{\Lambda}{|\mathbf{p}|} \right)^{2-\eta} \cdot \frac{1}{-i\Gamma^{-1}\Lambda^{-\eta}p_0/|\mathbf{p}|^{2-\eta} + 1}. \quad (39)$$

This yields the dynamic critical exponent

$$z = 2 - \eta, \quad (40)$$

which depends on N and d through η [63,66,67].

In order to better understand the result Eq. (40), let us carefully see how it appears. For that purpose, it is instructive to go back to the 1PI–NLO calculation, which, however, does not produce a dissipative mode in the IR region. The absence of a dissipative mode in 1PI–NLO is deduced from the IR behavior of $\Im \Sigma_R(p)$; namely, in the IR region it starts from $\mathcal{O}(p_0^2, \mathbf{p}^2)$ instead of $\mathcal{O}(p_0)$. Even though we do not have a dissipative mode, we are able to define a “dynamic exponent” from the dispersion relation in the IR region. However, this is an exponent for the “propagating mode” and must be distinguished from the exponent of a dissipative mode that governs the very long-time evolution of the system. Indeed, if one considers a two-point function in the 1PI–NLO calculation $[G_R^{1\text{PI-NLO}}(p)]^{-1} \sim -p_0^2 + \mathbf{p}^2 - \Re \Sigma_R^{1\text{PI-NLO}}(p) - i\Im \Sigma_R^{1\text{PI-NLO}}(p)$ with $\Im \Sigma_R^{1\text{PI-NLO}}(p) = \mathcal{O}(p_0^2, \mathbf{p}^2)$, one finds dispersions for “propagating modes” $p_0 \sim \pm |\mathbf{p}|$ as the leading contribution with a small correction from the self-energy of order $\mathcal{O}(1/N)$. Thus, one will find an exponent $z_{\text{prop.}} = 1 + \mathcal{O}(1/N)$, where we have introduced $z_{\text{prop.}}$ to remind that it is not the exponent for a dissipative mode, but for a propagating mode. If one performs the same analysis in *nonrelativistic* field theories, one will find $z_{\text{prop.}} = 2 + \mathcal{O}(1/N)$ again for propagating modes whose leading dispersions are $p_0 \propto \mathbf{p}^2$. Namely, the leading contribution z_0 in the exponent $z_{\text{prop.}} = z_0 + \mathcal{O}(1/N)$ depends on whether the system is relativistic or nonrelativistic, and this is one of the reasons why there was confusion in the literature. Now, coming back to the 2PI–NLO calculation where we do have a dissipative mode, we find that the presence of a dissipative term, $-i\Gamma^{-1}p_0$, modifies the dispersion, and the leading contribution to z is given by 2. Higher-order correction $-\eta$ to $z = 2$ is from $\Re \Sigma_R(p)$ and it is of $\mathcal{O}(1/N)$. Therefore, our result $z = 2 - \eta$ in Eq. (40) looks similar to the one for nonrelativistic propagating modes $z_{\text{prop.}} = 2 + \mathcal{O}(1/N)$, but the origin is different.

Let us further proceed one more step. Our result Eq. (40) is an immediate consequence of the static scaling behavior under the assumption that the relaxation term $-i\Gamma^{-1}p_0$ remains non-singular [68,69]. However, the relaxation constant Γ may depend on \mathbf{p} in general. It is explicitly shown in Eq. (12) of Ref. [17] that, as long as we stay in a region $|p_0|/|\mathbf{p}| \ll 1$ (see footnote 4), the self-energy at the critical point can be expanded as the power series of p_0 . Although the powers of p_0 are integers in this expansion, there appear $\ln \mathbf{p}$ terms at the critical point. Resumming the $\ln \mathbf{p}$ terms that are proportional to p_0 , one finds that the dissipative term depends on \mathbf{p} with the anomalous dimension $-c'$, such as $-i\Gamma^{-1}p_0 \sim -i\Gamma_0^{-1}(\Lambda/|\mathbf{p}|)^{c'}p_0$ with Γ_0 and c' being constants in the IR region. Then one obtains

$$G_R(p_0, \mathbf{p}) \sim \frac{1}{-i\Gamma_0^{-1}(\Lambda/|\mathbf{p}|)^{c'}p_0 + \Lambda^\eta \mathbf{p}^{2-\eta}}, \quad (41)$$

and accordingly the exponent z is modified to

$$z = 2 - \eta + c'. \quad (42)$$

This is also clearly seen in the time dependence of the two-point function. Equation (41) leads to $G_R(t, \mathbf{p}) \sim \exp\{-t/\tau(\mathbf{p})\}$ where the relaxation time is given by $\tau(\mathbf{p}) \sim 1/(\Gamma(\mathbf{p})p^{2-\eta}) \sim 1/p^{2-\eta+c'}$. Thus, in the IR region $p \rightarrow 0$, the relaxation time diverges. We have extended the evaluation of c' in Ref. [70].

Our result Eq. (42) with modified Γ is consistent with “model A” in the classification for nonrelativistic systems based on the mode-coupling theory [5]. Model A contains only (multi-component) non-conserved fields without conserved fields, and the dynamic critical exponent is given by $z = 2 + c\eta$, where c is a numerical constant of order one [5,17,71]. This seems apparently to be the case since we are only looking at the fundamental field φ . But it is actually nontrivial because in the 2PI framework the energy–momentum tensor is conserved and the coupling between the non-conserved fields φ and conserved fields is relevant to the dynamic universality classification. However, without knowing the explicit numerical value for c' , it is difficult at present to judge which model our result should be classified as.

There is an explicit numerical simulation for a particular case with $N = 1$ and $d = 2$ [16]. From numerical evaluation of the dynamic critical exponent $z \sim \frac{2-\eta}{0.87}$, it is claimed that it belongs to the dynamic universality class, model C. However, since we have used the $1/N$ expansion, which is not valid for $N = 1$, we cannot directly compare ours to the numerical simulation result, which is, however, intriguing.

5. Summary

The long-time behavior of a system recovering equilibrium should be characterized by a dissipative hydrodynamic mode. Because of the nonlinear mixing among the modes, we expect that even the two-point function G of the order-parameter field φ carries information on this long-time dissipation.

We have shown, however, that the NLO calculation of G in the 1PI $1/N$ expansion only gives real and imaginary corrections to the propagating mode of $\delta\varphi$, but does not generate the dissipative mode in G , and in particular that the imaginary part of the self-energy, $\Im\Sigma_R(p_0, \mathbf{p})/p_0$, vanishes in the IR limit, $p_0, \mathbf{p} \rightarrow 0$. At the critical point, we find that the mode spectrum scales as $p_0 \propto p^{z_{\text{prop.}}}$ with $z_{\text{prop.}} = z_0 + \mathcal{O}(1/N)$, where $z_0 = 1$ and 2 for relativistic and nonrelativistic cases, respectively. Since there is no dissipative mode in the IR limit, this exponent is just associated with the propagating mode, whose dispersion depends on whether the system is relativistic or nonrelativistic.

In the 2PI framework, on the other hand, we resummed the multiple scattering effects in the self-consistent two-point function G although working at NLO in the $1/N$ expansion. Once self-consistency is imposed on G , the spectral function $\rho(p_0, \mathbf{p})$ will become nonzero, except for $p_0 = 0$, even at NLO in the $1/N$ expansion. (Recall, e.g., the sunset diagram, which is included at this order.) We then argued that the dissipative mode inevitably emerges as an IR pole of the two-point function G by showing that the imaginary part of the self-energy $\Im\Sigma_R(p_0, \mathbf{p})/p_0$ is generally nonzero finite in the IR limit.

At the critical point, this dissipative mode implies that we will have $z \sim 2 - \eta$ even in relativistic scalar theories, which is in contrast to the 1PI–NLO result. For more precise evaluation of the exponent z , one may need to compute the transport coefficient utilizing the 4PI framework. We leave this as our future work. Furthermore, it is not clear whether the dissipative mode that we found is the same as the dissipation of the hydrodynamic mode, such as heat. To this end, we apparently need a more detailed and systematic evaluation of the coupled two-point functions of φ and the energy–momentum densities, etc.

Acknowledgements

The authors are very grateful to Jürgen Berges for useful discussions and correspondence on critical dynamics of $O(N)$ scalar theory. Y.S. and O.M. would like to thank Yoshimasa Hidaka for discussions on non-equilibrium phenomena in general. H.F.'s work was partially supported by a Grant-in-Aid for Scientific Research (C) No. 24540255.

Funding

Open Access funding: SCOAP³.

Appendix A. Derivation of Eq. (17)

In this appendix, we derive Eq. (17) in spatial d dimensions using the imaginary-time formalism. As we have graphically shown in Eq. (10), the (scattering part of the) self-energy at NLO is explicitly written as

$$\Sigma(i\omega_n, \mathbf{p}) = \frac{\lambda}{3N} \frac{\lambda}{6} T \sum_m \int \frac{d\mathbf{q}}{(2\pi)^d} G(i\omega_m, \mathbf{q}) D(i\omega_{n+m}, \mathbf{p} + \mathbf{q}), \quad (\text{A1})$$

where $i\omega_n$ is the Matsubara frequency, $i\omega_n = 2\pi i n / \beta$ ($n \in \mathbb{Z}$). In the 1PI formalism, G and D are *free* correlation functions, and summation over the Matsubara frequency is easily transformed into a contour integral on a complex plane of an analytically continued variable ζ . On the other hand, in the 2PI formalism, $G(i\omega_n, \mathbf{q})$ and $D(i\omega_n, \mathbf{q})$ are *full* correlation functions and, when analytically continued, both $G(\zeta, \mathbf{q})$ and $D(\zeta, \mathbf{q})$ have cuts along the real axis $\Im m \zeta = 0$, except for the origin $\zeta = 0$. Namely, $G(\zeta, \mathbf{q})$ and $D(\zeta, \mathbf{q})$ have discontinuities along the real axis, but one can define $G(0, \mathbf{q})$ and $D(0, \mathbf{q})$ without ambiguity. Thus, we have to be careful when we transform the Matsubara summation into a contour integral over the complex variable ζ [72,73]. This was done in Ref. [55] with a spatial dimension d of three. Here, we will follow Ref. [55], but work in d dimensions.

In the complex plane of ζ , one can recover the Matsubara summation by picking up all the residues of the function, $n(\zeta) = (e^{\beta\zeta} - 1)^{-1}$. Therefore, Eq. (A1) becomes

$$\Sigma(i\omega_n, \mathbf{p}) = \frac{\lambda}{3N} \frac{\lambda}{6} \int_{\mathcal{C}} \frac{d\zeta}{2\pi i} \int \frac{d\mathbf{q}}{(2\pi)^d} n(\zeta) G(\zeta, \mathbf{q}) D(i\omega_n + \zeta, \mathbf{p} + \mathbf{q}), \quad (\text{A2})$$

where the integral-contour \mathcal{C} is a collection of circles going counterclockwise around each pole of $n(\zeta)$: $\zeta = 2\pi i m / \beta$ ($m \in \mathbb{Z}$). We have to be careful when we pick up the poles at $\zeta = 0$ and $\zeta = -i\omega_n$ because there are cuts originating from the two-point functions, $G(\zeta)$ and $D(i\omega_n + \zeta)$. The easiest way to avoid this is to infinitesimally shift the poles so that one can safely pick up the residues. This simple prescription indeed works and the result does not depend on how to shift the poles.

Then, we modify the contour \mathcal{C} (the collection of small circles around the poles of $n(\zeta)$), paying attention¹² to the presence of cuts along $\zeta = 0$ and $\zeta = -i\omega_n$, and obtain a new contour \mathcal{C}' , as shown in Fig. A1.

In this integral, contributions from infinite distances become zero. Therefore, we pick up the integrals along two discontinuities. Namely, we consider four paths: from $\zeta = -\infty \pm i\epsilon$ to $\zeta = +\infty \pm i\epsilon$, and from $\zeta = -\infty - i\omega_n \pm i\epsilon$ to $\zeta = +\infty - i\omega_n \pm i\epsilon$ (ϵ is an infinitesimal positive

¹² $G(\zeta)$ has no poles in $\Im m \zeta \neq 0$ [43,49,74], and the same argument can be applied to $D(\zeta)$. Thus, we need not beware of singularities of $G(\zeta)$ and $D(\zeta)$, except for their discontinuities along the real axis.

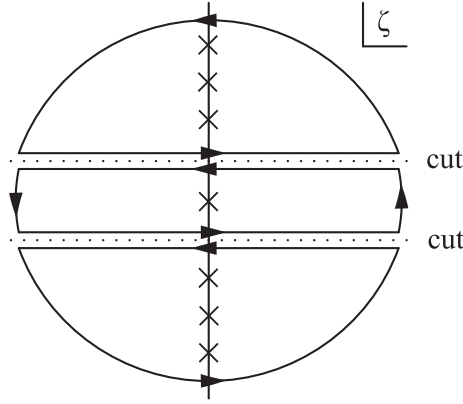


Fig. A1. The complex plane of ζ . Dotted lines along $\Im \zeta = 0$ and $\Im \zeta = -\omega_n$ represent cuts of $G(\zeta)$ and $D(i\omega_n + \zeta)$, respectively. The crosses denote poles of $n(\zeta)$, all of which are on the imaginary axis. The contour \mathcal{C} can be modified to three closed paths \mathcal{C}' (solid lines).

number). Thus, we find that Eq. (A2) becomes

$$\begin{aligned} \Sigma(i\omega_n, \mathbf{p}) &= \frac{\lambda}{3N} \frac{\lambda}{6} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi i} \int \frac{d\mathbf{q}}{(2\pi)^d} n(q_0) \\ &\quad \times \{ [G(q_0 + i\epsilon, \mathbf{q}) - G(q_0 - i\epsilon, \mathbf{q})] D(i\omega_n + q_0, \mathbf{p} + \mathbf{q}) \\ &\quad + G(q_0 - i\omega_n, \mathbf{q}) [D(q_0 + i\epsilon, \mathbf{p} + \mathbf{q}) - D(q_0 - i\epsilon, \mathbf{p} + \mathbf{q})] \}. \end{aligned} \quad (\text{A3})$$

In order to obtain the retarded self-energy, we perform an analytic continuation $i\omega_n \rightarrow p_0 + i\epsilon$:

$$\begin{aligned} \Sigma_R(p_0, \mathbf{p}) &= \frac{\lambda}{3N} \frac{\lambda}{6} \int \frac{dq_0 d\mathbf{q}}{(2\pi)^{d+1}} n(q_0) \\ &\quad \times \{ \rho(q_0, \mathbf{q}) D_R(p_0 + q_0, \mathbf{p} + \mathbf{q}) + G_A(q_0 - p_0, \mathbf{q}) \rho_D(q_0, \mathbf{p} + \mathbf{q}) \}, \end{aligned} \quad (\text{A4})$$

where we have used the following properties:

$$\rho(q_0, \mathbf{q}) = -i [G_R(q_0, \mathbf{q}) - G_A(q_0, \mathbf{q})], \quad \rho_D(q_0, \mathbf{q}) = -i [D_R(q_0, \mathbf{q}) - D_A(q_0, \mathbf{q})]. \quad (\text{A5})$$

Now, we take the imaginary part of Eq. (A4). Since the spectral functions are imaginary parts of correlation functions,

$$\rho(p) = 2 \Im G_R(p) = -2 \Im G_A(p), \quad \rho_D(p) = 2 \Im D_R(p) = -2 \Im D_A(p), \quad (\text{A6})$$

we finally obtain Eq. (17):

$$\begin{aligned} \Im \Sigma_R(p_0, \mathbf{p}) &= \frac{\lambda}{6N} \frac{\lambda}{6} \int \frac{dq_0 d\mathbf{q}}{(2\pi)^{d+1}} n(q_0) \\ &\quad \times \{ \rho(q_0, \mathbf{q}) \rho_D(p_0 + q_0, \mathbf{p} + \mathbf{q}) - \rho(q_0 - p_0, \mathbf{q}) \rho_D(q_0, \mathbf{p} + \mathbf{q}) \} \\ &= \frac{\lambda}{6N} \frac{\lambda}{6} \int \frac{dq_0 d\mathbf{q}}{(2\pi)^{d+1}} \rho(q) \rho_D(p + q) [n(q_0) - n(p_0 + q_0)]. \end{aligned} \quad (\text{A7})$$

In the second line of Eq. (A7), we have changed the integration variables of the second term to $q_0 \rightarrow q_0 + p_0$ and $\mathbf{q} \rightarrow \mathbf{p} + \mathbf{q}$.

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